Math 126 HW 1

2.2) A.) $\frac{\partial}{\partial r} = \frac{\partial x_1}{\partial r} \cdot \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial r} \cdot \frac{\partial}{\partial x_2}$

$$= \cos(\theta)\frac{\partial}{\partial x_1} + \sin(\theta)\frac{\partial}{\partial x_2}$$

$$\frac{\partial}{\partial \theta} = (-r\sin\theta)\frac{\partial}{\partial x_1} + (r\cos(\theta))\frac{\partial}{\partial x_2}$$

B.) $\frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1} \cdot \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x_1} \cdot \frac{\partial}{\partial \theta}$

$r = \sqrt{x_1^2 + x_2^2} \Rightarrow \frac{\partial r}{\partial x_1} = \frac{x_1}{\sqrt{x_1^2+x_2^2}} = \frac{r\cos(\theta)}{r}$

$\theta = \arctan(x_2/x_1) \Rightarrow \frac{\partial \theta}{\partial x_1} = \left(\frac{1}{1+(x_2/x_1)^2}\right)\left(\frac{-x_2}{(x_1)^2}\right)$

$$= \frac{-x_2}{x_1^2+x_2^2} = \frac{-\sin(\theta)}{r}$$

$$\frac{\partial}{\partial x_1} = \cos(\theta)\frac{\partial}{\partial r} + \left(-\frac{\sin(\theta)}{r}\right)\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x_1^2} = \cos^2(\theta)\frac{\partial^2}{\partial r^2} - \sin(\theta)\cos(\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial}{\partial r}\right) + \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\left(\frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial x_2} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x_2^2} = \sin^2(\theta)\frac{\partial^2}{\partial r^2} + \sin(\theta)\cos(\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial r}\right) + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\left(\frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial r}\right) - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial}{\partial r}\right)$$
$$+ \frac{\sin(\theta)}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\sin(\theta)}{1}\frac{\partial}{\partial \theta}\right) + \frac{\cos(\theta)}{r^2}\frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial}{\partial \theta}\right)$$

$$\frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial r}\right) = \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2}{\partial \theta \partial r} + \frac{\cos(\theta)}{r}(\cos(\theta))\frac{\partial}{\partial r}$$

and ~~the~~ expanding all of these gives

$$\Delta = \partial^2/\partial r^2 + 1/r\ \partial/\partial r + 1/r^2\ \partial^2/\partial \theta^2$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + 1/r^2\ \partial^2/\partial \theta^2$$

Optional: 2.4

3.1) $\begin{cases} \dot{x}(t) = c \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = x_0 + ct$; $\frac{Du}{Dt} + 0 = 0 \Rightarrow$

$$u(t, x(t)) = \begin{cases} g(x_0) & \text{if } x_0 \geq 0 \\ h(x_0/c) & \text{if } x_0 \leq 0 \end{cases}$$

or

$$u(t,x) = \begin{cases} g(x-ct) & x \geq ct \\ h(x/c - t) & x \leq ct \end{cases}$$

B.) We need that $\frac{\partial}{\partial t} g(x-ct)$ evaluated at $x = ct$ is the same as $\frac{\partial}{\partial t} h(x/c - t)$ at the same line, and the piecewise function $\partial_t u = \begin{cases} g'(x-ct)(-c) & x > ct \\ h'(x/c - t)(-1) & x < ct \end{cases}$ is continuous.

i.e. $\lim_{s \to 0} -cg'(s) = \lim_{s \to 0} -h'(s)$

Similarly, $\partial_x u = \begin{cases} g'(x-ct) & x > ct \\ \frac{1}{c} h'(x/c - t) & x < ct \end{cases}$

gives $\lim_{s \to 0} g'(s) = \lim_{s \to 0} \frac{1}{c} h'(s)$ (the same condition)

Lastly, $g(0) = h(0)$.

Homework 1 Solutions

3.2) A.) $x(t) = x_0 + ct$

$$\begin{cases} \dfrac{Du}{Dt} - f(t, x(t)) = 0 \\ u(0, x_0) = g(x_0) \end{cases} \Rightarrow \quad u(t, x(t)) = g(x_0) + \int_0^t f(s, x(s))\,ds$$

So

$$u(t, x) = g(x - ct) + \int_0^t f(s, x + c(s-t))\,ds$$

B.) $x(t) = x_0 + ct$

$$\begin{cases} \dfrac{Du}{Dt} = r(t, x(t))\,u(t, x(t)) \\ u(0, x_0) = g(x_0) \end{cases}$$

$$\Rightarrow \ln|u| = \int_0^t r(\xi, x(\xi))\,ds + c$$

$$u(t, x(t)) = g(x_0)\, e^{\int_0^t r(s, x(s))\,ds}$$

$$u(t, x) = g(x - ct)\, e^{\int_0^t r(s, x + c(s-t))\,ds}$$

3.6) B) We create a system

$$\begin{cases} \dot{x}(t) = u(t, x(t)) \\ x(0) = x_0 \end{cases} \qquad \begin{cases} \frac{Du}{Dt} = 0 \\ u(0, x(0)) = u(0, x_0) \end{cases}$$

So $u(t, x(t)) = u(0, x_0) = a(1-x_0) + b(x_0)$ for $x_0 \in [0,1]$

$x(t) = x_0 + t(a(1-x_0) + b(x_0))$

and $x\left(\frac{1}{a-b}\right) = -a/a-b$.